

①

Hermitian operators

An operator $\hat{A}$ having two eigenfunctions ψ_1 and ψ_2 is said to be Hermitian if

$$\int \psi_1 (\hat{A} \psi_2) d\tau = \int \psi_2 (\hat{A} \psi_1) d\tau$$

When ψ_1 and ψ_2 are real.

$$\text{OR, } \int \psi_1^* (\hat{A} \psi_2) d\tau = \int \psi_2 (\hat{A} \psi_1)^* d\tau$$

When ψ_1 and ψ_2 are complex. Here ψ_1^* is the complex conjugate of ψ_1 and $d\tau$ is the volume element of space in which the functions ψ_1 and ψ_2 are defined.

The functions $e^{ix} (\psi_1)$ and $\sin x (\psi_2)$ are the two acceptable eigenfunctions of the operator $\frac{d^2}{dx^2} (\hat{A})$.

NOW,

$$\int \psi_1^* (\hat{A} \psi_2) d\tau = \int e^{-ix} \left(\frac{d^2}{dx^2} \sin x \right) dx$$

$$= - \int e^{-ix} \sin x dx \quad \text{--- (1)}$$

and,

$$\int \psi_2 (\hat{A} \psi_1)^* d\tau = \int \sin x \cdot \left[\frac{d^2 (e^{ix})^*}{dx^2} \right] dx$$

$$= \int \sin x (i^2 e^{ix})^* dx$$

$$= - \int \sin x e^{-ix} dx \quad \text{--- (2)}$$

from eqn (1) and (2)

$$\int \psi_1^* (\hat{A} \psi_2) d\tau = \int \psi_2 (\hat{A} \psi_1)^* d\tau$$

Hence, the operator $\hat{A} = \frac{d^2}{dx^2}$ is a

Hermitian operator.

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The eigen values of a Hermitian operator are real. $\Rightarrow$

Let,

$\hat{A}$ be a Hermitian operator with eigen function ψ and corresponding eigen value a .

Now, we have

$$\hat{A} \psi = a \psi \quad \text{--- (1)}$$

and

$$(\hat{A} \psi)^* = a^* \psi^* \quad \text{--- (2)}$$

Multiplying the eqn (1) with ψ^* and integrating, we have

$$\int \psi^* \hat{A} \psi d\tau = \int \psi^* a \psi d\tau$$

$$= a \int \psi^* \psi d\tau \quad \text{--- (3)}$$

Again multiplying the eqⁿ. (2) with ψ and then integrating, we get

$$\int \psi (\hat{A} \psi)^* d\tau = \int \psi a^* \psi^* d\tau$$

$$= a^* \int \psi^* \psi d\tau \quad \text{--- (4)}$$

Since $\hat{A}$ is Hermitian operator,

$$\int \psi^* \hat{A} \psi d\tau = \int \psi (\hat{A} \psi)^* d\tau \quad \text{--- (5)}$$

from eqⁿ. (3), (4) and (5), we have

$$a \int \psi \psi^* d\tau = a^* \int \psi \psi^* d\tau$$

or,

$$a = a^*$$

It is possible only when a is real.

Thus, a Hermitian operator has real eigenvalues.