Hermitian operators

An operator $\hat{A}$ having two eigenfunctions ψ_1 and ψ_2 is said to be Hermitian if

$$\int \psi_1 (\hat{A}\psi_2)\, d\tau = \int \psi_2 (\hat{A}\psi_1)\, d\tau$$

When ψ_1 and ψ_2 are real.

or,
$$\int \psi_1^* (\hat{A}\psi_2)\, d\tau = \int \psi_2 (\hat{A}\psi_1)^*\, d\tau$$

When ψ_1 and ψ_2 are complex. Here ψ_1^* is the complex conjugate of ψ_1 and $d\tau$ is the volume element of space in which the functions ψ_1 and ψ_2 defined.

The functions e^{ix} (ψ_1) and $\sin x$ (ψ_2) are the two acceptable eigenfunction of the operator $\frac{d^2}{dx^2}$ $(\hat{A})$,

Now,
$$\int \psi_1^* (\hat{A}\psi_2)\, d\tau = \int e^{-ix} \left(\frac{d^2}{dx^2} \sin x\right) dx$$

$$= -\int e^{-ix} \sin x \, dx \quad \text{--- (1)}$$

and,
$$\int \psi_2 (A\psi_1)^*\, d\tau = \int \sin x \cdot \left[\frac{d^2 (e^{ix})^*}{dx^2}\right] dx$$

$$= \int \sin x \, (i^2 e^{ix})^*\, dx$$

$$= -\int \sin x \, e^{-ix} \, dx \quad \text{———— (2)}$$

from eqn (1) and (2)

$$\int \psi_1^* (\hat{A} \psi_2) d\tau = \int \psi_2 (A \psi_1)^* d\tau$$

Hence, the operator $\hat{A} = \frac{d^2}{dx^2}$ is a Hermitian operator.

* The eigen values of a Hermitian operators are real. $\Rightarrow$

Let,

$\hat{A}$ be a Hermitian operator with eigen function ψ and corresponding eigen value a.

Now, we have

$$\hat{A}\psi = a\psi \quad \text{———— (1)}$$

and

$$(\hat{A}\psi)^* = a^* \psi^* \quad \text{———— (2)}$$

Multiplying the eqn (1) with ψ^* and integrating, we have

$$\int \psi^* \hat{A} \psi d\tau = \int \psi^* a \psi d\tau$$

$$= a \int \psi^* \psi \, d\tau \quad \text{———— (3)}$$

Again multiplying the eqn. (2) with ψ and then integrating, we get

$$\int \psi (\hat{A}\psi)^* d\tau = \int \psi a^* \psi^* d\tau$$

$$= a^* \int \psi^* \psi \, d\tau \quad \text{——— (4)}$$

Since $\hat{A}$ is Hermitian operator,

$$\int \psi^* \hat{A} \psi \, d\tau = \int \psi (\hat{A}\psi)^* d\tau \quad \text{——— (5)}$$

from eqn. (3), (4) and (5), we have

$$a \int \psi \psi^* d\tau = a^* \int \psi \psi^* d\tau$$

or,

$$a = a^*$$

It is possible only when a is real.

Thus, a Hermitian operator has real eigen values.