

Weierstrass Approximation Theorem

Statement:

Let f be a continuous real-valued function on a closed and bounded interval $[a,b]$. Then for every $\varepsilon > 0$, there exists a polynomial $P(x)$ such that $|f(x) - P(x)| < \varepsilon$ for all x in $[a,b]$.

Proof:

It is sufficient to prove the theorem for the interval $[0,1]$. Any closed interval $[a,b]$ can be reduced to $[0,1]$ by a linear transformation.

Define the Bernstein polynomial of degree n by $B_n(f;x) = \sum_{k=0}^n f(k/n) C(n,k) x^k (1-x)^{n-k}$, $k = 0$ to n .

Each $B_n(f;x)$ is a polynomial in x . Since f is continuous on $[0,1]$, it is uniformly continuous. Hence for every $\varepsilon > 0$, there exists $\delta > 0$ such that $|x - y| < \delta$ implies $|f(x) - f(y)| < \varepsilon$.

As $n \rightarrow \infty$, Bernstein polynomials converge uniformly to f on $[0,1]$. Therefore, $|B_n(f;x) - f(x)| < \varepsilon$ for all x in $[0,1]$ and sufficiently large n .

Thus, every continuous function on a closed interval can be uniformly approximated by polynomials.

Hence proved.