(a) commutative property of operators

(b) commutator operator

(c) linear operator.

Ans. (a) Commutative property of operators : If two oeprators re such that the result of their successive operations is the same respective of the order of their operations, then the two operators re said to be commutative.

For the two operators $\hat{A}$ and $\hat{B}$ to be commutative,

$$\hat{A}\hat{B}f(x) = \hat{B}\hat{A}f(x)$$

Let, $\hat{A}$ and $\hat{B}$ be $\frac{d}{dx}$ and $\frac{d^2}{dx^2}$ respectively, and $f(x) = \sin x$.

Now,
$$\hat{A}\hat{B}f(x) = \frac{d}{dx}\left[\frac{d^2}{dx^2} . \sin x\right]$$

$$= \frac{d}{dx}(-\sin x)$$

$$= -\cos x$$

$$\hat{B}\hat{A}\, f(x) = \frac{d^2}{dx^2}\left[\frac{d}{dx}\sin x\right]$$

$$= \frac{d^2}{dx^2} \cdot \cos x$$

$$= -\cos x$$

Here, $\hat{A}$ and $\hat{B}$ are said to commute.

The significance of the commutating operators lies in the fact that the observables corresponding to their eigen values can be determined precisely and simultaneously.

(b) Commutator operator : If two operators $\hat{A}$ and $\hat{B}$ do not commute, then,

$\hat{A}\hat{B} - \hat{B}\hat{A}$, is called commutator operator.

The commutator operator, $\hat{A}\hat{B} - \hat{B}\hat{A}$, is also designated as $[\hat{A}, \hat{B}]$.

Let $\hat{A} = \hat{p}_x$ (linear momentum operator) and $\hat{B} = \hat{x}$ (position operator) and a function be $f(x)$. Then,

$$(\hat{p}_x\hat{x} - \hat{x}\hat{p}_x) f(x) = \frac{h}{2\pi i}\frac{d}{dx}[x \,.\, f(x)] - x \,.\, \frac{h}{2\pi i}\frac{d}{dx} f(x)$$

$$= \frac{h}{2\pi i}\left[x \,.\, \frac{d\,(f_x)}{dx} + f(x)\right] - x . \frac{h}{2\pi i}\, f'(x)$$

$$= \frac{h}{2\pi i}[x \,.\, f'(x) + f(x)] - \frac{h}{2\pi i} \cdot x + f'(x)$$

$$= \frac{h}{2\pi i} f(x)$$

Thus, the commutator, $(\hat{p}_x \,.\, \hat{x} - \hat{x} \,.\, \hat{p}_x) = \frac{h}{2\pi i}$

This means that the operators $\hat{p}_x$ and $\hat{x}$ do not commute and hence it is not possible to determine the precise values of the position and momentum of a particle simultaneously, quite in line with the Heisenberg's uncertainty principle.

(c) **Linear operator** : An operator $\hat{A}$ is said to be linear if its application on the sum of two functions gives the result which is equal to the sum of the operations on the two functions separately, i.e.,

$$\hat{A}\,[f(x) + g(x)] = \hat{A}\, f(x) + \hat{A}\, g(x)$$

or, $\hat{A} \,.\, Cf(x) = C.\,\hat{A}\, f(x)$, C = constant.

The operator $\frac{d}{dx}$ is linear operator because,

$$\frac{d}{dx}(ax^m + bx^n) = \frac{d}{dx}(ax^m) + \frac{d}{dx}(bx^n).$$

On the other hand, $\sqrt{\ }$ (square root), is not a linear operator because.

$$\sqrt{g(x) + f(x)} \neq \sqrt{g(x)} + \sqrt{f(x)}\,.$$