and Prove Cauchy's root test theorem.

be always positive and $\lim_{n\to\infty} u_n^{1/n} = l$

(i) If $l<1$, Σu_n is Convergent

(ii) If $l>1$, Σu_n is divergent

(iii) If $l=1$, the test fails and the series may b either Convergent or divergent.

let $l<1$

Since $l<1$, we can choose and $\epsilon>0$ such that

$l+\epsilon<1$

let $l+\epsilon=r$ so that $0<r<1$

Since $\lim_{n\to\infty} u_n^{1/n} = l$ therefore there exists positive integer m such that

$|u_n^{1/n} - l| < \epsilon$ when $n \geq m$.

i.e $l-\epsilon < u_n^{1/n} < l+\epsilon$ when $n \geq m$

i.e $(l-\epsilon)^n < u_n < (l+\epsilon)^n$ when $n \geq m$

Taking the last inequality we find that

$u_n < r^n$ where $r = l+\epsilon < 1$ for all $n \geq m$.

But the series $r^m + r^{m+1} + \cdots$ is a series in G.P. whose common ratio $r<1$ and hence it is Convergent. Therefore by the Comparison test

$u_m + u_{m+1} + \cdots$ is Convergent

Hence Σu_n is Convergent

(ii) let $l>1$

Since $l>1$, we can choose ~~and~~ an $\epsilon>0$ such that

$l-\epsilon>1$

$n \to \infty$

Such that

$$|u_n^{1/n} - l| < \epsilon \quad \text{when } n \geq m$$

i.e $\quad l - \epsilon < u_n^{1/n} < l + \epsilon \quad$ when $n \geq m$

i.e $\quad (l - \epsilon)^n < u_n < (l + \epsilon)^n \quad$ when $n \geq m$

Taking the first inequality, we find that

$$u_n > R^n \quad \text{where } R = l - \epsilon > 1 \text{ for all } n \geq m.$$

Thus we find that u_n does not tend to zero. Hence Σu_n is not convergent. But a series of positive terms must either converge or diverge and so Σu_n is divergent.

Case III. We will illustrate this case when $l = 1$ by taking suitable examples.

Let $u_n = \frac{1}{n}$ then $u_n^{1/n} = \left(\frac{1}{n}\right)^{1/n} = \frac{1}{n^{1/n}}$

$$\therefore \lim_{n \to \infty} u_n^{1/n} = \frac{1}{\lim_{n \to \infty} n^{1/n}} = \frac{1}{1} = 1$$

But $\Sigma \frac{1}{n}$ is a divergent series

Thus we see in this case that when $\lim u_n^{1/n} = 1$ the series Σu_n is divergent

on the other hand let $u_n = \frac{1}{n^2}$

then in this case also $u_n^{1/n} = \frac{1}{n^{2/n}} = \frac{1}{(n^{1/n})^2}$

$$\therefore \lim_{n \to \infty} u_n^{1/n} = 1$$

But the series $\Sigma \frac{1}{n^2}$ is a convergent series

Thus we see that in this case when $\lim_{n \to \infty} u_n^{1/n} = 1$ the series Σu_n is convergent.

These two examples show that if $\lim_{n \to \infty} u_n^{1/n} = 1$ the series Σu_n may be either convergent or divergent that is the test fails