Important Problem of cyclic groups.

Pro. 1. Show that the group $(G, \times_7)$ is cyclic, Where $G = \{1,2,3,4,5,6\}$. How many generators are there?

Solution: — Firstly we shall prove that if $\exists$ an element $a \in G$. such that $o(a) = 6 = o(G)$, then G will be a cyclic group and a will be the generator of G.
If e is the identity in G then $e = 1$.

Observe that $3^1 = 3$, $3^2 = 3 \times_7 3 = 2$
For $3 \times 3 = 9 = 1 \times 7 + 2$

$3^3 = 3^2 \times_7 3 = 2 \times_7 3 = 6$.

$3^4 = 3^3 \times_7 3 = 6 \times_7 3 = 4$

$3^5 = 3^4 \times_7 3 = 4 \times_7 3 = 5$. For $4 \times 3 = 12 = 1 \times 7 + 5$

$3^6 = 3^5 \times_7 3 = 5 \times_7 3 = 1$ For $5 \times 3 = 15 = 2 \times 7 + 1$.

Thus $3^6 = e$ and $3^r \neq e$ for $r < 4$.

$\therefore$ This $\Rightarrow o(3) = 6 = o(G)$
$\Rightarrow 3$ is a generator of G.

Since $3^6 = 1$, $3^5 = 5$, $3^4 = 4$, $3^3 = 6$, $3^2 = 2$, $3^1 = 3$
Hence G is expressible as
$G = \{3^6, 3^5, 3^4, 3^3, 3^2, 3\}$.

This shows that G is cyclic

Now, we are to determine the number of generators of G.

If d is H.C.F of m and n, then we write $(m, n) = d$

An element $3^m \in G$ is also a generator of G if $(m, 6) = 1$.

Evidently $(1, 6) = 1$, $(5, 6) = 1$.

Hence there are only two generators of G namely $3, 3^5$.

Ans.

Problem — 02 How many generators are there of the cyclic group of order 8?

Solution: — Let a cyclic group G of order 8 generated by an element a, then

$o(a) = o(G) = 8$.

To determine the number of generators of G.

Evidently $G = \{a, a^2, a^3, a^4, a^5, a^6, a^7, a^8 = e$

An element $a^m \in G$ is also a generator of G is H.C.F of m and 8 is 1.

H.C.F of 1 and 8 is 1, H.C.F. of 3 and 8 is 1, H.C.F. of 5 and 8 is 1, H.C.F of 7 and 8 is 1.

Hence a, a^3, a^5, a^7, are generators of G.

Therefore, there are four generators of

Ans.

Problem — 03. Prove that the group $[\{1, -1, i, -i\}, \cdot$ is cyclic.

Solution: — Let $G = \{1, -1, i, -i\}$.

To prove that $(G, \cdot)$ is cyclic.

If $\exists$ an element $a \in G$ s.t. $o(a) = 4 = o(G$ then G will be a cyclic group with its generator a.

Evidently $i^1 = i$, $i^2 = -1$, $i^3 = -i$, i^4

here identity element e of G is 1.

Thus $i^4 = 1$, $i^r \neq 1$ for any $r <$

this $\Rightarrow o(i) = 4 = o(G)$

$\Rightarrow i$ is the generator of

Now G is expressible as $G = \{i, i^2, i^3, i^4$

This shows that G is cyc