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## **COMPLEX ANALYSIS (PG LEVEL NOTES)**

### **1. Mittag-Leffler Theorem**

Statement: Let  $\{a_n\}$  be a sequence of distinct complex numbers with no finite limit point. For each  $n$ , let  $P_n(z)$  be a principal part at  $a_n$ . Then there exists a meromorphic function  $f(z)$  having poles precisely at  $a_n$  with prescribed principal parts  $P_n(z)$ . Idea of Proof: We construct functions  $f_n(z) = P_n(z) - Q_n(z)$ , where  $Q_n(z)$  are suitable polynomials chosen so that the series  $\sum f_n(z)$  converges uniformly on compact subsets. Uniform convergence ensures analyticity away from poles. The resulting function is meromorphic and has required principal parts. Sketch of Proof: 1. Construct  $P_n(z)$  near each pole. 2. Subtract suitable polynomial  $Q_n(z)$  to control growth. 3. Show  $\sum (P_n(z) - Q_n(z))$  converges normally. 4. The resulting function is meromorphic with desired poles.

Example 1: Construct a meromorphic function with simple poles at integers with residue 1. Consider  $f(z) = \sum [1/(z-n) - 1/n]$  over  $n \neq 0$ . This converges normally and defines required function.  
Example 2: Construct function with principal part  $1/(z-a)^2$  at  $a$ . Then  $f(z) = 1/(z-a)^2$  is meromorphic with double pole at  $a$ .

## 2. Argument Principle

Statement: Let  $f$  be meromorphic inside and on a simple closed contour  $C$ , with no zeros or poles on  $C$ . Then  $\frac{1}{2\pi i} \oint_C \frac{f'(z)}{f(z)} dz = N - P$ , where  $N$  = number of zeros,  $P$  = number of poles (counted with multiplicity). Proof: Let zeros be  $a_k$  and poles be  $b_j$ . Then near each zero of order  $m$ ,  $f(z) = (z - a_k)^m g(z)$ ,  $g(a_k) \neq 0$ . Hence  $f'/f$  has simple pole with residue  $m$ . Similarly each pole contributes residue  $-m$ . Using residue theorem, the integral equals sum of residues =  $N - P$ .

Example 1: Count zeros of  $f(z) = z^3 - 1$  inside  $|z|=2$ . Since polynomial has 3 zeros and no poles, Argument principle gives  $N = 3$ . Example 2: Count zeros of  $f(z) = z^4 + 5z + 1$  inside  $|z|=1$ . Compare boundary behavior or compute change of argument numerically.

### 3. Rouché's Theorem

Statement: Let  $f$  and  $g$  be analytic inside and on simple closed contour  $C$ . If  $|g(z)| < |f(z)|$  on  $C$ , then  $f$  and  $f+g$  have same number of zeros inside  $C$ . Proof: Consider  $F(z) = f(z) + g(z)$ . On  $C$ ,  $|g| < |f|$  implies  $f$  and  $F$  never vanish on  $C$ . Consider homotopy  $H(z,t)=f(z)+t g(z)$ . Number of zeros remains constant as  $t$  varies from 0 to 1. By argument principle,  $f$  and  $F$  have same number of zeros.

Example 1: Count zeros of  $z^5 + 3z + 1$  inside  $|z|=1$ . On  $|z|=1$ ,  $|z^5|=1$  and  $|3z+1| \leq 4$ . Instead compare  $|3z|=3 > |z^5+1| \leq 2$ . Hence  $3z$  dominates; one zero inside. Example 2: Show  $z^4 + 10z + 5$  has exactly one zero inside  $|z|=1$ . On  $|z|=1$ ,  $|10z|=10 > |z^4+5| \leq 6$ . Hence same number of zeros as  $10z$ , i.e., one zero.