

# Dr. Anupama Sinha

*Assistant Professor, H.D. Jain College, Ara*

## Reduction of Second-Order Linear Partial Differential Equations to Canonical Form

### 1. Introduction

A general second-order linear partial differential equation (PDE) in two variables  $x$  and  $y$  is:

$$A(x,y) u_{xx} + 2B(x,y) u_{xy} + C(x,y) u_{yy} + D(x,y) u_x + E(x,y) u_y + F(x,y) u = G(x,y)$$

The main aim of reduction to canonical form is to simplify the second-order part of the equation

by transforming the variables so that the mixed derivative term disappears or the equation

takes a simpler standard form.

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### 2. Classification of Second-Order Linear PDE

Consider only the second-order terms:

$$A u_{xx} + 2B u_{xy} + C u_{yy}$$

Compute the discriminant:

$$\Delta = B^2 - AC$$

Case 1:  $\Delta > 0 \rightarrow$  Hyperbolic

Case 2:  $\Delta = 0 \rightarrow$  Parabolic

Case 3:  $\Delta < 0 \rightarrow$  Elliptic

This classification determines the nature of transformation required.

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### 3. Method of Reduction to Canonical Form

Step 1: Identify A, B, C.

Step 2: Compute  $\Delta = B^2 - AC$ .

Step 3: Solve the auxiliary (characteristic) equation:

$$A(dy/dx)^2 - 2B(dy/dx) + C = 0$$

Step 4: Solve for  $dy/dx$  to obtain characteristic curves.

Step 5: Introduce new variables  $\xi$  and  $\eta$  along characteristic curves.

Step 6: Rewrite the PDE in terms of  $\xi$  and  $\eta$ .

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#### 4. Examples

Example 1 (Hyperbolic):

$$u_{xx} - u_{yy} = 0$$

Here  $A=1$ ,  $B=0$ ,  $C=-1$

$$\Delta = 0 - (1)(-1) = 1 > 0 \text{ (Hyperbolic)}$$

Characteristic equation:

$$(dy/dx)^2 - 1 = 0$$

$$dy/dx = \pm 1$$

Integrating:

$$y - x = \text{constant}$$

$$y + x = \text{constant}$$

Let:

$$\xi = y - x$$

$$\eta = y + x$$

The equation reduces to:

$$u_{\xi\eta} = 0 \text{ (Canonical form)}$$

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Example 2 (Parabolic):

$$u_{xx} + 2u_{xy} + u_{yy} = 0$$

$$A=1, B=1, C=1$$

$$\Delta = 1 - 1 = 0 \text{ (Parabolic)}$$

Characteristic equation:

$$(dy/dx - 1)^2 = 0$$

$$dy/dx = 1$$

Characteristic curve:

$$y - x = \text{constant}$$

Canonical form reduces to:

$$u_{\xi\xi} = 0$$

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Example 3 (Elliptic):

$$u_{xx} + u_{yy} = 0 \text{ (Laplace equation)}$$

$$A=1, B=0, C=1$$

$$\Delta = 0 - 1 = -1 < 0 \text{ (Elliptic)}$$

Using transformation:

$$\xi = x$$

$$\eta = y$$

Already in canonical form:

$$u_{\xi\xi} + u_{\eta\eta} = 0$$

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Example 4:

$$4u_{xx} - 4u_{xy} + u_{yy} = 0$$

$$A=4, B=-2, C=1$$

$$\Delta = 4 - 4 = 0 \text{ (Parabolic)}$$

Characteristic equation:

$$4m^2 + 4m + 1 = 0$$

$$(2m + 1)^2 = 0$$

$$m = -1/2$$

Characteristic curve:

$$2y + x = \text{constant}$$

Canonical form reduces to:

$$u_{\xi\xi} = 0$$

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Example 5:

$$u_{xx} + 4u_{xy} + 3u_{yy} = 0$$

$$A=1, B=2, C=3$$

$$\Delta = 4 - 3 = 1 > 0 \text{ (Hyperbolic)}$$

Characteristic equation:

$$m^2 - 4m + 3 = 0$$

$$(m - 1)(m - 3) = 0$$

$$m = 1, 3$$

Characteristic curves:

$$y - x = \text{constant}$$

$$y - 3x = \text{constant}$$

Canonical form:

$$u_{\xi\eta} = 0$$

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Example 6:

$$9u_{xx} + 12u_{xy} + 4u_{yy} = 0$$

$$A=9, B=6, C=4$$

$$\Delta = 36 - 36 = 0 \text{ (Parabolic)}$$

Characteristic equation:

$$9m^2 - 12m + 4 = 0$$

$$(3m - 2)^2 = 0$$

$$m = 2/3$$

Characteristic curve:

$$3y - 2x = \text{constant}$$

Canonical form:

$$u_{\xi\xi} = 0$$

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## 5. Conclusion

Reduction to canonical form simplifies second-order PDEs and helps in solving them easily.

The discriminant  $\Delta = B^2 - AC$  plays a crucial role in classification.

Hyperbolic equations reduce to  $u_{\xi\eta}$  form,

Parabolic equations reduce to  $u_{\xi\xi}$  form,

Elliptic equations reduce to  $u_{\xi\xi} + u_{\eta\eta}$  form.