Urysohn's Lemma and Tietze Extension Theorem

1. Urysohn's Lemma

Statement:
Let X be a normal topological space and let A and B be two disjoint closed subsets of X.
Then there exists a continuous function $f : X \to [0,1]$ such that:
$f(x) = 0$ for all x in A and
$f(x) = 1$ for all x in B.

Explanation:
A topological space is called normal if any two disjoint closed sets can be separated by disjoint open sets. Urysohn's lemma strengthens this property by showing that we can construct a continuous function that separates the two closed sets numerically.

Example:
Consider X = R with the usual topology.
Let A = [0,1] and B = [2,3]. These are disjoint closed sets.
Define a function:

$f(x) = 0$ for $x \leq 1$
$f(x) = (x - 1)$ for $1 \leq x \leq 2$
$f(x) = 1$ for $x \geq 2$

This function is continuous and satisfies the required properties:
f = 0 on A and f = 1 on B.
Thus, Urysohn's lemma holds in R.

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2. Tietze Extension Theorem

Statement:
Let X be a normal topological space and A be a closed subset of X.
If $f : A \to R$ is continuous and bounded, then there exists a continuous extension $F : X \to R$ such that $F(x) = f(x)$ for all x in A.

Explanation:
The theorem guarantees that any continuous function defined on a closed subset of a normal space can be extended to the whole space without losing continuity.

Example:
Let X = R and A = [0,1].
Define $f(x) = x^2$ on A.

Since f is continuous and bounded on [0,1], by Tietze extension theorem, there exists a continuous function F on R such that $F(x) = x^2$ for x in [0,1].

One simple extension is:
$F(x) = x^2$ for all x in R.

This clearly extends f and remains continuous on R.

Conclusion:
Urysohn's lemma provides the foundation for constructing continuous functions that separate closed sets, while the Tietze extension theorem allows continuous functions defined on closed subsets to be extended to the entire space.
Both results are fundamental in topology and functional analysis.