

Branches of Many-Valued Functions

Dr. Anupama Sinha

Assistant Professor, Department of Mathematics
HD Jain College, Ara

1. Introduction

In Complex Analysis, several functions such as argument, logarithm, and power functions are many-valued. To make them single-valued, we define appropriate branches using branch cuts. A branch of a many-valued function is a single-valued analytic function obtained by restricting the domain suitably.

2. Argument of z ($\arg z$)

If $z = re^{i\theta}$, then θ is called the argument of z . Since $\theta + 2n\pi$ ($n \in \mathbb{Z}$) also represents the same complex number, $\arg z$ is many-valued. Principal value: $\text{Arg } z \in (-\pi, \pi]$. Branch cut usually taken along the negative real axis.

Examples: 1. $\arg(1 + i) = \pi/4 + 2n\pi$ 2. $\arg(-1) = \pi + 2n\pi$ 3. $\arg(i) = \pi/2 + 2n\pi$ 4. Principal value $\text{Arg}(-1 - i) = -3\pi/4$ 5. $\arg(re^{i\pi/3}) = \pi/3 + 2n\pi$

3. Logarithm Function ($\log z$)

$\log z = \log r + i(\theta + 2n\pi)$, where $z = re^{i\theta}$. Thus $\log z$ is infinitely many-valued. Principal branch: $\text{Log } z = \ln r + i \text{Arg } z$, with branch cut along negative real axis.

Examples: 1. $\log(1) = 2n\pi i$ 2. $\log(i) = i(\pi/2 + 2n\pi)$ 3. $\text{Log}(-1) = i\pi$ (principal value) 4. $\log(e^{i\pi}) = i(\pi + 2n\pi)$ 5. $\log(2e^{i\pi/4}) = \ln 2 + i(\pi/4 + 2n\pi)$ 6. $\text{Log}(1+i) = (1/2)\ln 2 + i\pi/4$

4. Bilinear (Möbius) Transformation

A bilinear transformation is defined as: $w = (az + b)/(cz + d)$, where $ad - bc \neq 0$. Properties: • Maps circles and lines into circles or lines. • Preserves cross-ratio. • Conformal except where derivative vanishes. • One-to-one mapping on extended complex plane.

Examples: 1. $w = 1/z$ maps circles through origin to lines. 2. $w = (z-1)/(z+1)$ maps right half-plane to unit disk. 3. $w = z+1$ translates plane. 4. $w = 2z$ rotates and dilates. 5. $w = (z-i)/(z+i)$ maps upper half-plane to unit circle. 6. $w = (2z+3)/(z+4)$

5. Classification of Branches

Branches are classified based on: • Choice of branch cut • Principal branch • Finite-valued branches (e.g., n -th root) • Infinite-valued branches (logarithm) • Analytic continuation

6. Conclusion

Many-valued functions arise naturally in complex analysis. By defining branches carefully, we convert them into analytic single-valued functions. Understanding $\arg z$, $\log z$, and bilinear transformations is fundamental for advanced topics like conformal mapping and Riemann surfaces.